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Supersymmetric mechanics with an odd action functional

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Abstract. We give an odd Lagrangian formulation of models yielding the Poisson–Buttin bracket in the graded phase space. Such a generalization of the usual supersymmetric mechanics allows the introduction of an n (for n-extended algebra, $n = 2k$) even parameters of deformation of the geometry of an n-extended super-time space.

1. Introduction

Supersymmetric mechanical systems are typically considered within the framework in which the Hamiltonian of the system is an even element of the observable algebra. However, as was pointed out in [1, 2], there exists another possibility. The algebra of phase space functions can be given by means of the Buttin bracket [2]. In contrast to the Poisson bracket, the Buttin bracket is an odd mapping and has shifted grade properties (for a general description of such super-Hamiltonian structures see [3]). An odd bracket of this kind is also widely used in the Hamiltonian BRST formalism and in that context is called an antibracket (for a review of the BRST antibracket formalism see [4]). Recall that the odd mechanics can be considered independently of the antifield BRST formalism [3].

The first examples of Hamiltonian models realizing an odd bracket mechanics and quantization procedure were presented in the series of papers [5–8]. These models were defined directly in a graded phase space by means of an odd Hamiltonian, but no Lagrangian description has yet been given.

In the present paper, starting from the configuration space description, we show that even and odd models can be treated as two different grade realizations of conventional $n = 2k$ supersymmetry. In the case of odd models, we obtain an n-parameter family of such models with an appropriately deformed antisymmetric form on an n-extended super-time space. The parity shift mapping plays an important role here [9, 10]. Its role for the graded supersymmetric mechanics was observed in [11].

2. Supersymmetric classical mechanics with an odd action functional

Let us consider the superfield supersymmetric classical mechanics (cf [11]) $(Y; \{Q_\alpha, D_\alpha, \partial\}; (M, J, \overset{s}{G}); S)$ of a dimension (N_0, N_1) with $N_0 = N_1$. Where:

(a) Y is the complex super-time space, $(t, \vartheta_\alpha) \in Y$; $\alpha = 1, 2, \ldots, n$ (for convenience $n = 2$), where t is commuting and ϑ_α are anticommuting variables.

(b) $\{Q_\alpha, D_\alpha, \partial\}$ is the set of generators and covariant derivatives of the $n = 2$ supersymmetry algebra, i.e.

$$[Q_\alpha, Q_\beta] = 2g_{\alpha\beta}\mathrm{i}\partial \qquad [D_\alpha, D_\beta] = -2g_{\alpha\beta}\mathrm{i}\partial \tag{1a}$$

$$[Q_\alpha, \partial] = 0 \qquad [D_\alpha, \partial] = 0 \tag{1b}$$

$$[Q_\alpha, D_\beta] = 0 \tag{1c}$$

($[\cdot, \cdot]$ is a graded Lie bracket and $g_{\alpha\beta}$ a non degenerate symmetric metric cf appendix). In addition we have the generator R of $O(2)$ rotations, which are automorphisms of this superalgebra

$$[R, Q_\alpha] = c_\alpha{}^\beta Q_\beta. \tag{2}$$

Realization of the above operators is conventional:

$$\partial = \frac{\mathrm{d}}{\mathrm{d}t} \qquad Q_\alpha = \partial_\alpha + \mathrm{i}\vartheta_\alpha\partial \qquad D_\alpha = \partial_\alpha - \mathrm{i}\vartheta_\alpha\partial \qquad R = -\vartheta_\alpha c^{\alpha\beta}\partial_\beta. \tag{3}$$

(c) A graded configuration space $M = M_0 + M_1$ ($\dim M_s = N_s, s = 0, 1$) is endowed with the graded symmetric metric, i.e. for $\phi = (\overset{0}{\phi}, \overset{1}{\phi}) \in M$ we have

$$\langle \overset{s}{\phi}, \overset{s}{\phi} \rangle = \sum \overset{s}{G}{}^{ij} \overset{s}{\phi}_i \overset{s}{\phi}_j \qquad \overset{s}{G}_{ij} = (-)^s \overset{s}{G}_{ji} \qquad \langle \overset{s}{\phi}, \overset{s'}{\phi} \rangle = 0 \qquad \text{for } s \neq s'. \tag{4}$$

Therefore trajectories in M are described by functions

$$\overset{0}{\phi}_j(t, \vartheta) \equiv X_j(t) = x_j(t) + \mathrm{i}\vartheta_\alpha x_j^\alpha(t) + \tfrac{1}{2}\vartheta^2 b_j(t) \tag{5a}$$

$$\overset{1}{\phi}_j(t, \vartheta) \equiv Y_j(t) = y_j(t) + \vartheta_\alpha y_j^\alpha(t) + \tfrac{1}{2}\vartheta^2 f_j(t). \tag{5b}$$

The component functions of the above coordinate superfunctions describe the motion of a model in extended configuration superspace of dimension $2^n 2N$. Here $\overset{0}{\phi}$ and $\overset{1}{\phi}$ have vectorial index, however, in other types of supersymmetric mechanics the index of the odd superfield can be spinorial (for example in the superfield spinning superparticle model [12]).

(d) S denotes the action of the supersymmetric system. For an even mechanics it is an even functional (in the sense of the Grassmann parity, $\deg(S_0) = 0$). We shall generalize it here to the case $\deg(S_1) = 1$. To this end let us start with the kinetic term of the S_0. Generally it is of the form

$$S_0 = \tfrac{1}{4}\int \mathrm{d}t\, \mathrm{d}\vartheta_2\, \mathrm{d}\vartheta_1\, c^{\alpha\beta}\langle D_\alpha\phi, D_\beta\phi\rangle. \tag{6}$$

The configuration space is a direct sum of odd and even sectors therefore D_α is understood here as $D_\alpha \otimes \mathrm{id}_M$. Naturally, the above action is invariant under the supersymmetry transformations generated by Q_α, ∂ and also under rotations generated by R (the last one acts non-trivially on components x_j^α, y_j^α). We neglect here the target space symmetries.

New possibilities give an odd extension of covariant derivative operators. Namely, using the parity shift operator Π [9–11] we can introduce the even derivative $D_\alpha \otimes \Pi$, acting on the graded vector superfunction. It is natural to assume that Π^2 is an identity and allow Π to be different for each D_α. Therefore in the case of $n = 2$, matrix realization of such a parity shift operator will be simply (generalization to the $n = 2k$ is obvious)

$$\Pi_{q_\alpha} = \begin{pmatrix} 0 & q_\alpha^{-1} \\ q_\alpha & 0 \end{pmatrix} \qquad \alpha = 1, 2 \tag{7}$$

where q_α is an invertible even parameter (q_α can be complex). Having such an extension we can finally define the odd action functional as follows:

$$S_1 = \tfrac{1}{2} \int \mathrm{d}t\, \mathrm{d}\vartheta_2\, \mathrm{d}\vartheta_1\, \langle D_\alpha \phi, \Pi^{\alpha\beta} D_\beta \phi \rangle = \int \mathrm{d}t\, L_1 \tag{8}$$

where

$$\Pi^{\alpha\beta} D_\beta \overset{0}{\phi}_i = c^{\alpha\beta} q_\beta^{-1} D_\beta \overset{1}{\phi}_i \qquad \text{and} \qquad \Pi^{\alpha\beta} D_\beta \overset{1}{\phi}_i = c^{\alpha\beta} q_\beta D_\beta \overset{0}{\phi}_i \tag{9}$$

This means, in the notation of (5), that

$$\begin{aligned} \overset{s}{S}_1 &= \tfrac{1}{2} \int \mathrm{d}t\, \mathrm{d}\vartheta_2\, \mathrm{d}\vartheta_1\, \overset{s}{G}{}^{ij}\, \overset{s}{c}{}_q^{\alpha\beta} D_\alpha X_i D_\beta Y_j \\ &= \tfrac{1}{2} \int \mathrm{d}t\, \overset{s}{G}{}^{ij} \left(\mathrm{Tr}\, \overset{s}{\delta}_q (\dot{x}_i \dot{y}_j - b_i f_j) + \tfrac{1}{2} (\overset{s}{\delta}{}^{\alpha}_{q\beta} x_{\alpha i} \dot{y}^{\beta}_j - \overset{s}{\delta}_{q\alpha}{}^{\beta} \dot{x}^{\alpha}_i y_{j\beta}) \right) \end{aligned} \tag{10}$$

where

$$\overset{s}{c}{}_q^{\alpha\beta} = \begin{pmatrix} 0 & -(q_1^{-1} - (-)^s q_2) \\ q_2^{-1} - (-)^s q_1 & 0 \end{pmatrix} \tag{11}$$

(for other notational conventions, see the appendix).

The form $\overset{s}{c}{}_q^{\alpha\beta}$ is the deformation of the antisymmetric form $c^{\alpha\beta}$, different for each sector s ($s = 0, 1$) of the configuration space. It depends on two parameters $(q_1, q_2) \equiv q$, $\alpha = 1, 2$ (recall that we assume $n = 2$). Deformed $c^{\alpha\beta}$ is not antisymmetric in general and indicates changes in the geometry of super-time space Y.

The action S_1 (like S_0) is invariant under supersymmetry transformations and rotations. However, there is a difference in the behaviour of the Lagrangian L_1 in comparison with L_0. Namely, under infinitesimal rotation it changes by a total time derivative (analogous variation of L_0 vanishes). Explicitly in components this divergence term has the form

$$\delta \overset{s}{L} = -\delta\varphi \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\overset{s}{G}{}^{ij} \left(\overset{s}{c}{}_q^{\alpha\beta} + \overset{s}{c}{}_q^{\beta\alpha} \right) x_{\alpha i} y_{\beta j} \right). \tag{12}$$

Our discussion up to now has concerned the kinetic term only. Let us consider a more interesting system possessing a potential term as well.

The so-called graded superfield oscillator (GSO) was introduced in [11] as a supersymmetric system containing the full set of bosonic and fermionic oscillators and rotators. Here, we will generalize the GSO to the odd case. First, let us note that deformation present in the kinetic term of odd models is strictly connected with the presence of the covariant derivative operators, therefore it will not be present in the potential term. We define the action of the odd GSO (OGSO) in the following form:

$$S=\int \mathrm{d}t\,\mathrm{d}\vartheta_2\,\mathrm{d}\vartheta_1\,(\tfrac{1}{2}\langle D_\alpha\phi,\Pi^{\alpha\beta}D_\beta\rangle-\omega\langle\phi,\Pi\phi\rangle)$$

$$=\int \mathrm{d}t\sum_{s=1,2}\overset{s}{G}_{ij}\Big\{\tfrac{1}{2}\Big[\mathrm{Tr}\,\overset{s}{\delta}_q(\dot{x}_i\dot{y}_j-b_if_j)+\tfrac{1}{2}\Big(\overset{s}{\delta}{}^{\alpha}_{q\beta}x_{\alpha i}\dot{y}^{\beta}_j-\overset{s}{\delta}_{q\alpha}{}^{\beta}\dot{x}^{\alpha}_i y_{\beta j}\Big)\Big]$$

$$+(-)^s\omega(x_if_j+ic_{\alpha\beta}x^{\alpha}_iy^{\beta}_j+b_iy_j)\Big\}. \tag{13}$$

The above action yields the following equations of motion:

$$\tfrac{1}{2}\,\mathrm{Tr}\,\overset{s}{\delta}_q\ddot{x}_i-(-)^s\omega b_i=0 \tag{14a}$$

$$\tfrac{1}{4}\Big(\overset{s}{\delta}{}^{\beta}_{q\alpha}+\overset{s}{\delta}_{q\alpha}{}^{\beta}\Big)\dot{x}_{\beta i}+(-)^s\mathrm{i}\omega c_{\alpha\beta}x^{\beta}_i=0 \tag{14b}$$

$$\tfrac{1}{2}b_i-(-)^s\omega x_i=0 \tag{14c}$$

and

$$\tfrac{1}{2}\,\mathrm{Tr}\,\overset{s}{\delta}_q\ddot{y}_i-(-)^s\omega f_i=0 \tag{15a}$$

$$\tfrac{1}{4}\Big(\overset{s}{\delta}{}^{\beta}_{q\alpha}+\overset{s}{\delta}_{q\alpha}{}^{\beta}\Big)\dot{y}_{\beta i}+(-)^s\mathrm{i}\omega c_{\alpha\beta}y^{\beta}_i=0 \tag{15b}$$

$$\tfrac{1}{2}f_i-(-)^s\omega y_i=0. \tag{15c}$$

Eliminating auxiliary components we can finally write these equations in the form:

$$\tfrac{1}{2}\,\mathrm{Tr}\,\overset{s}{\delta}_q\ddot{x}_i+\omega^2x_i=0 \tag{16a}$$

$$\tfrac{1}{2}\,\mathrm{Tr}\,\overset{s}{\delta}_q\dot{x}_{\alpha i}+(-)^s\mathrm{i}\omega c_{\alpha\beta}x^{\beta}_i=0 \tag{16b}$$

$$\tfrac{1}{2}\,\mathrm{Tr}\,\overset{s}{\delta}_q\ddot{y}_i+\omega^2y_i=0 \tag{17a}$$

$$\tfrac{1}{2}\,\mathrm{Tr}\,\overset{s}{\delta}_q\dot{y}_{\alpha i}+(-)^s\mathrm{i}\omega c_{\alpha\beta}y^{\beta}_i=0. \tag{17b}$$

Therefore the odd GSO contains the system of bosonic oscillators and rotators (equations (16a), (17b)) and the system of fermionic oscillators and rotators (equations (17a), (16b)). On the level of the equations of motion the deformation can be absorbed to mass of the oscillator and rotator. Moreover for $q_1=q_2=(\mathrm{i})^{s+1}$, $\overset{s}{c}_{\alpha\beta}\sim c_{\alpha\beta}$; $\overset{s+1}{c}{}_{\alpha\beta}=0$. In this case the whole system projects on the subspace $(M_s,\overset{s}{G})$. Note that all expressions where s enters are considered mod 2. For $q_1\neq q_2$ both sectors of the odd GSO model are present.

3. Hamiltonian description: the Dirac–Buttin algebra of charges

The model introduced in section 2 is supersymmetric. Therefore, performing the passage to the graded phase space we expect that the system will have constraints. Because of the odd character of the Lagrangian, the resulting Hamiltonian will be an odd function. Obviously the Legendre transformation gives a Hamiltonian with the same parity as that of the original Lagrangian. Parity of this transformation is the same as the parity of the time parameter. In the present models it is fixed to be even.

Canonical momenta have opposite grade with respect to the grade of the conjugated coordinates:

$$\overset{s}{p}{}^i_x = \tfrac{1}{2}\,\mathrm{Tr}\,\overset{s}{\delta}_q\,\overset{s}{G}{}^{ij}\dot{y}_j \tag{18a}$$

$$\overset{s}{p}{}^i_{x\alpha} = -\tfrac{1}{2}\,\overset{s}{G}{}^{ij}\,\overset{s}{\delta}_{q\alpha}{}^{\beta}y_{\beta j} \tag{18b}$$

$$p^i_b = 0 \tag{18c}$$

and

$$\overset{s}{p}{}^i_y = (-)^s\tfrac{1}{2}\,\mathrm{Tr}\,\overset{s}{\delta}_q\,\overset{s}{G}{}^{ij}\dot{x}_j \tag{19a}$$

$$\overset{s}{p}{}^i_{y\alpha} = (-)^s\tfrac{1}{2}\,\overset{s}{G}{}^{ij}\,\overset{s}{\delta}{}^{\beta}_{q\alpha}x_{\beta j} \tag{19b}$$

$$p^i_f = 0 \tag{19c}$$

where $\deg F = \deg p_F + 1$; F is an arbitrary coordinate and p_F is its momentum. Expressions (18*b*), (19*b*) and (18*c*), (19*c*) yield the primary second-class constraints. The last relations allows one to get rid of auxiliary degrees of freedom: b_i, f_j. Non-trivial constraints are taken into account in the following definitions:

$$\overset{s}{G}{}^i_{x\alpha} \equiv p^i_{x\alpha} + \tfrac{1}{2}\,\overset{s}{G}{}^{ij}\,\overset{s}{\delta}_{q\alpha}{}^{\beta}y_{\beta j} = 0 \tag{20a}$$

$$\overset{s}{G}{}^i_{y\alpha} \equiv p^i_{y\alpha} - (-)^s\tfrac{1}{2}\,\overset{s}{G}{}^{ij}\,\overset{s}{\delta}_q{}^{\beta}{}_{\alpha}x_{\beta j} = 0. \tag{20b}$$

Finally, the Hamiltonian in the reduced phase space is of the form

$$H_q = \left(\tfrac{1}{2}\,\mathrm{Tr}\,\overset{s}{\delta}_q\right)^{-1}\overset{s}{G}_{ij}\,\overset{s}{p}{}^i_y\,\overset{s}{p}{}^j_x \qquad \deg H = 1. \tag{21}$$

It can be shown that the Poisson–Buttin (PB) bracket for odd system has the following form (for reduced form see [2] and also [5–8])

$$\{A, B\}_1 = \sum_r (-)^{r(A+1)}\left(\frac{\partial A}{\partial F_r}\frac{\partial}{\partial p^r_F} + (-)^A\frac{\partial A}{\partial p^r_F}\frac{\partial}{\partial F_r}\right)B \tag{22}$$

where $\deg F_r = r$ and external indices are omitted; A, B are the phase space functions. The BP bracket is connected with the so-called periplectic form defined on the graded phase space (cf [3]). Because of the mixed parity of F and p_F this bracket is an odd mapping, i.e. $\deg(\{A, B\}_1) = \deg A + \deg B + 1$. Algebraically, it has the properties of the Buttin bracket [1–3] (cf also [5, 7]). It is the graded bracket with grading factors changed by the grade of the bracket itself. Namely,

$$\{A, B\}_1 = -(-)^{(A+1)(B+1)}\{B, A\}_1 \tag{23a}$$

$$\sum_{\text{cycl}}(-)^{(A+1)(C+1)}\{A, \{B, C\}_1\}_1 = 0 \tag{23b}$$

$$\{A, BC\}_1 = \{A, B\}_1 C - (-)^{(A+1)B} B\{A, C\}_1 \tag{23c}$$

In particular $\{,\}_1$ is symmetric only when $\deg A = \deg B = 0$. For the canonical variables we obtain that

$$\{F^i, p_{Fj}\}_1 = (-)^{r(r+1)}\delta^i_j = \delta^i_j \tag{24}$$

where $\deg F^i = r$ and $\deg p^i_F = r+1$. In contrast to the even Hamiltonian graded mechanics the minus sign is not present here for $r = 1$.

Because of the constraints (15) we have to modify the relations (22) to the case of the Dirac–Buttin bracket. Namely, the resulting Dirac–Buttin bracket is

$$\{A, B\}_1^{*,s} = \{A, B\}_1 + \left(\tfrac{1}{2}\operatorname{Tr}\overset{s}{\delta}_q\right)^{-1} \times \left((-)^s \left\{A, \overset{s}{G}{}^i_{x\alpha}\right\}_1 \overset{s}{G}{}^{-1}_{ij} \left\{\overset{s}{G}{}^{\alpha j}_y, B\right\}_1 - \left\{A, \overset{s}{G}{}^i_{y\alpha}\right\}_1 \overset{s}{G}{}^{-1}_{ij} \left\{\overset{s}{G}{}^{\alpha j}_x, B\right\}_1\right). \tag{25}$$

Finally, essential relations for canonical variables have the form:

$$\left\{x_i, \overset{s}{p}{}^j_x\right\}_1^{*,s} = \overset{s}{\delta}{}^j_i \qquad \left\{y_i, \overset{s}{p}{}^j_y\right\}_1^{*,s} = \overset{s}{\delta}{}_i{}^j \tag{26a}$$

$$\left\{x_{\alpha i}, y_{\beta j}\right\}_1^{*,s} = -(-)^s \left(\tfrac{1}{2}\operatorname{Tr}\overset{s}{\delta}_q\right) g_{\alpha\beta}\, \overset{s}{G}{}^{-1}_{ij}. \tag{26b}$$

Conserved charges for the considered model read, in phase space, as follows:

$$\overset{s}{Q}{}^\alpha_q = \mathrm{i}x^\alpha_i \overset{s}{p}{}^i_x + y^\alpha_i \overset{s}{p}{}^i_y \qquad \deg Q = 0 \tag{27a}$$

$$\overset{s}{R}_q = -\left(\tfrac{1}{2}\operatorname{Tr}\overset{s}{\delta}_q\right) \overset{s}{G}{}^{ij} c^{\alpha\beta} x_{\alpha i} y_{\beta j} \qquad \deg R = 1. \tag{27b}$$

Together with the Hamiltonian (21) they form a closed Dirac–Buttin algebra with the basic relations

$$\left\{\overset{s}{Q}{}^\alpha_q\, \overset{s}{Q}{}^\beta_q\right\}_1^{*,s} = 2\mathrm{i}g^{\alpha\beta} H_q \tag{28}$$

$$\left\{\overset{s}{R}_q\, \overset{s}{Q}{}^\alpha_q\right\}_1^{*,s} = c^\alpha_\beta\, \overset{s}{Q}{}^\beta_q. \tag{29}$$

Therefore for each fixed s (connected with the geometric structure G^s_{ij} in the target space) we obtain the Dirac–Buttin algebra anti-realization of $n = 2$ supersymmetry. This anti-realization is parametrized by two even parameters q_1, q_2. Prefix 'anti' means here that the grade of the deformed realization of each generator is shifted by one with respect to the former grade (cf equation (1)). The multiplication in the above algebra has shifted-grade properties.

4. Final remarks

Let us look briefly at the relations between these two realizations of the supersymmetric mechanics. Note, that we have conventional supersymmetry algebra in both cases. Formally we work within the diagram

$$\begin{array}{ccccc} TM & \xrightarrow{FL_0} & T_0^*M & \longleftarrow & F_0 \\ \downarrow \Pi_{TM} & & \downarrow \Pi_{T^*M} & & \downarrow \Pi_F \\ T_\pi M & \xrightarrow[FL_1^q]{} & T_1^*M & \longleftarrow & F_1 \end{array} \tag{30}$$

where: M denotes the configuration superspace, TM is the tangent superspace for the even system, $T_\pi M$ is the tangent superspace for the odd system, i.e. with the Grassmann odd-valued 'scalar' product $\langle , \rangle_\pi = \langle , \Pi \rangle$ (here the deformation of the supertime structure also enters). The T_r^*M; $r = 0, 1$, are phase spaces of the even and odd systems, respectively; F_r are the Poisson and the Poisson–Buttin algebras of phase space functions for $r = 0, 1$, respectively. The Lagrangian L_0 and the q-family of Lagrangians L_1 define appropriate Legendre transformations FL_r, $r = 0, 1$. Since the form of parity shift on TM is fixed (cf definition of L_1 in (9)) as well as both Legendre transformations, introducing the mapping Π^*_{TM}, we can make the left part of the diagram commutative. It identifies momenta in both spaces. In particular, the anti-isomorphism between the algebras of constants of motion in F_1 and F_0 can be realized using the adjoint action with a fixed generating function. For the considered model with the kinetic term we have

$$\{E_\pi^1, \}_1^* : F_1 \longrightarrow F_0^\pi \tag{31}$$

where deg $E_\pi^1 = 0$ and

$$E_\pi^1 = -y_i \overset{1}{p}{}_x^i + x_i \left(\overset{0}{G} \left(\overset{1}{G}{}^T \right)^{-1} \right)^i_k \overset{1}{p}{}_y^k. \tag{32}$$

Moreover F_0^π is the algebra of functions of the form $f_0^\pi = f_0(F, \Pi_*(p_F))$.

Appendix

We use the following conventions:

$$c_{\alpha\beta} = -c_{\beta\alpha} \qquad c_{12} = -1 \qquad c_{\alpha\gamma} c^{-1\gamma\beta} = \delta_\alpha^\beta \tag{A1}$$

$$g_{\alpha\beta} = g_{\beta\alpha} \qquad g_{\alpha\gamma} g^{\gamma\beta} = \delta_\alpha^\beta \qquad c^{\alpha\beta} = g^{\alpha\gamma} c_{\gamma\rho} g^{\rho\beta} = -c^{-1\alpha\beta} \tag{A2}$$

$$\vartheta^\alpha = g^{\alpha\beta} \vartheta_\beta \qquad \tilde{\vartheta}^\alpha = c^{-1\alpha\beta} \vartheta_\beta \qquad \vartheta_\alpha = g_{\alpha\beta} \vartheta^\beta \qquad \tilde{\vartheta}_\alpha = c_{\alpha\beta} \vartheta^\beta \qquad c_\alpha^\beta = c_{\alpha\rho} g^{\rho\beta} \tag{A3}$$

$$\tilde{\vartheta}_\alpha = -g_{\alpha\beta} \tilde{\vartheta}^\beta \qquad \tilde{\vartheta}^\alpha = -g^{\alpha\beta} \tilde{\vartheta}_\beta \qquad \vartheta_\alpha = c_{\alpha\beta} \tilde{\vartheta}^\beta \qquad \vartheta^\alpha = c^{-1\alpha\beta} \tilde{\vartheta}_\beta \tag{A4}$$

$$\vartheta^2 = c_{\alpha\beta} \vartheta^\alpha \vartheta^\beta \qquad \vartheta_\alpha \vartheta_\beta = \tfrac{1}{2} c_{\alpha\beta} \vartheta^2 \qquad \partial_\alpha \doteq \frac{\partial}{\partial \vartheta} \alpha \qquad \partial_\alpha \vartheta^\beta = \delta_\alpha^\beta \tag{A5}$$

$$c_q^{\alpha\beta} c_{\beta\gamma} \equiv -\delta_{q\,\gamma}^{\alpha} \qquad c_{\alpha\gamma} c_q^{\gamma\beta} \equiv -\delta_{q\alpha}{}^{\beta} \tag{A6}$$

$$\mathrm{Tr}\, \overset{s}{\delta}_q = \mathrm{Tr} \left(\overset{s}{\delta}{}_{q\alpha}^{\beta} \right) = \mathrm{Tr} \left(\overset{s}{\delta}_{q\alpha}{}^{\beta} \right) = \overset{s}{q}_{12} + \overset{s}{q}_{21} \tag{A7}$$

where $\overset{s}{q}_{\alpha\beta} = q_\alpha - (-)^s q_\beta$.

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